A Proofs of Selected Results

Proposition 4. *Formula-based incision functions and standard incision functions are interchangeable, that is,*

1. *for all standard incision function σ there is a formula-based incision function σ' such that $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\alpha)$, for all α;*
2. *for all formula-based incision function σ' there is a standard incision function σ such that $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\alpha)$, for all α;*

Proof. Let us fix an arbitrary belief base $\mathcal{K}$.

1. Let σ be a standard incision function on $\mathcal{K}$, we construct the following formula-based function σ':
$$\sigma'(\alpha) = \sigma(\mathcal{K}\perp\!\!\!\perp \alpha).$$
It is clear that σ and σ' coincide and that σ' is a well-defined function.
We still need to prove that σ' is indeed a formula-based function. Satisfaction of conditions (1) and (2) in Definition 3 follows respectively from conditions (1) and (2) in Definition 2. For item (3), note that if two formulae α and β have the same set of kernels, then $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma(\mathcal{K}\perp\!\!\!\perp \beta)$ which implies $\sigma'(\alpha) = \sigma'(\beta)$.
2. Let σ' be a formula-based incision function on $\mathcal{K}$, we construct the following standard-incision function σ:
$$\sigma(X) = \sigma'(\alpha), \text{ for some } \alpha \in \mathcal{L}, \text{ such that } X = \mathcal{K}\perp\!\!\!\perp \alpha.$$
We have to show three things: (a) σ is a well-defined function, (2) σ is a standard-incision function, and (c) $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\alpha)$, for all $\alpha \in \mathcal{L}$. For item (a), let $X \in \mathcal{C}(\mathcal{K})$. Thus, $X = \mathcal{K}\perp\!\!\!\perp \beta$, for some $\beta \in \mathcal{L}$. This concludes the proof that σ is a well-defined function. We proceed to show that σ is a standard incision-function (b) and that $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\alpha)$, for all $\alpha \in \mathcal{L}$ (c). Let $\alpha \in \mathcal{L}$. From construction
$$\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\beta),$$
for some $\beta \in \mathcal{L}$ such that
$$\mathcal{K}\perp\!\!\!\perp \alpha = \mathcal{K}\perp\!\!\!\perp \beta.$$
Let us fix such a β. To prove that σ is indeed a standard-incision function we have to show that items (1) and (2) from Definition 2 are satisfied:
(1) we will show that $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) \subseteq \bigcup \mathcal{K}\perp\!\!\!\perp \alpha$. From item (1) at Definition 3, we get that $\sigma'(\beta) \subseteq \bigcup \mathcal{K}\perp\!\!\!\perp \beta$. Thus, as $\mathcal{K}\perp\!\!\!\perp \alpha = \mathcal{K}\perp\!\!\!\perp \beta$ and $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\beta)$, we get $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) \subseteq \bigcup \mathcal{K}\perp\!\!\!\perp \alpha$.
(2) let $X \in \mathcal{K}\perp\!\!\!\perp \alpha$ such that $X \neq \emptyset$. We will show that $X \cap \sigma(\mathcal{K}\perp\!\!\!\perp \alpha) \neq \emptyset$. As $\mathcal{K}\perp\!\!\!\perp \alpha = \mathcal{K}\perp\!\!\!\perp \beta$, we get that $X \in \mathcal{K}\perp\!\!\!\perp \beta$. Thus, from item (2) at Definition 3, we get that $X \cap \sigma'(\beta) \neq \emptyset$. Thus, as $\sigma'(\beta) = \sigma(\mathcal{K}\perp\!\!\!\perp \alpha)$, we get that $X \cap \sigma(\mathcal{K}\perp\!\!\!\perp \alpha) \neq \emptyset$.
This concludes the proof that σ is a standard-incision function (b). We proceed to show (c) $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\alpha)$. From condition (3) of Definition 3, we have that $\sigma'(\alpha) = \sigma'(\beta)$. This jointly with $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\beta)$ implies that $\sigma(\mathcal{K}\perp\!\!\!\perp \alpha) = \sigma'(\alpha)$. □

Theorem 7. *If Cn is Tarskian and satisfies compactness, then a contraction function satisfies* success*,* inclusion*,* vacuity*,* uniformity*,* core-retainment*, and* relative-closure *iff it is a smooth kernel contraction function.*

Proof. Let Cn be a Tarskian consequence operator satisfying compactness.

"⇒" Let $\dot{-}$ be a contraction function, defined on a belief base $\mathcal{K}$, satisfying *success*, *inclusion*, *vacuity*, *uniformity*, *core-retainment*, and *relative-closure*. We have from Theorem 6 that $\dot{-}$ corresponds to a kernel contraction function $\dot{-}_\sigma$. To complete the proof, we need to show that σ is smooth. Let $\alpha, \varphi \in \mathcal{L}$ and $\mathcal{K}' \subseteq \mathcal{K}$ such that $\varphi \in Cn(\mathcal{K}')$ and $\varphi \in \sigma(\alpha)$. Thus, $\varphi \in \mathcal{K}$. We will show that $\mathcal{K}' \cap \sigma(\alpha) \neq \emptyset$. Let us suppose, for contradiction, that $\mathcal{K}' \cap \sigma(\alpha) = \emptyset$. Thus, $\mathcal{K}' \subseteq \mathcal{K} \dot{-}_\sigma \alpha$, which implies from monotonicty that $\varphi \in Cn(\mathcal{K} \dot{-}_\sigma \alpha)$. As $\dot{-}_\sigma$ satisfies *relative-closure*, we have that $\varphi \in \mathcal{K} \dot{-}_\sigma \alpha$. This implies that $\varphi \notin \sigma(\alpha)$ which is a contradiction. Hence, σ satisfies smoothness.

"⇐" let $\dot{-}_\sigma$ be a smooth contraction function, defined on a belief base $\mathcal{K}$. We already have from Theorem 6 that in the presence of monotonicity and compactness, kernel contractions are characterised by the first five rationality postulates. So, we only need to show that $\dot{-}_\sigma$ satisfies *relative-closure*. Let us suppose for contradiction that it does not satisfy relative-closure. Thus, there are formulae α and β such that $\beta \in \mathcal{K}$, $\beta \in Cn(\mathcal{K} \dot{-}_\sigma \alpha)$ but $\beta \notin \mathcal{K} \dot{-}_\sigma \alpha$. Recall from definition of contraction functions that $\mathcal{K} \dot{-}_\sigma \alpha = \mathcal{K} \setminus \sigma(\alpha)$. Therefore,
$$(\mathcal{K} \dot{-}_\sigma \alpha) \cap \sigma(\alpha) = \emptyset,$$
$\mathcal{K} \dot{-}_\sigma \alpha \subseteq \mathcal{K}$ and $\beta \in \sigma(\alpha)$. By hypothesis $\beta \notin \mathcal{K} \dot{-}_\sigma \alpha$. Thus, from smoothness, we have $(\mathcal{K} \dot{-}_\sigma \alpha) \cap \sigma(\alpha) \neq \emptyset$, which is a contradiction. Hence, we conclude that $\dot{-}_\sigma$ satisfies relative-closure.

□

Observation A.1. *If Cn is compact than every α-kernel is finite.*

Proof. Let $\mathcal{K}$ be a belief base, and and α-kernel $A \in \perp\!\!\!\perp \alpha$, for some formula α. As A is an α-kernel, it entails α. Thus, As Cn is compact, there is a $A' \subseteq A$ such that $\alpha \in Cn(A')$. However, as A is an α-kernel, there is no proper subset of A that entails α, which means that $A \not\subset A$. Therefore, $A = A'$ which means that A is finite. □

Lemma A.2. *Given an α-shard $\leqslant_\alpha$ on a belief base $\mathcal{K}$ such that $\alpha \in Cn(\mathcal{K})$. For every $X \subseteq \mathcal{K}$, if X is finite and non-empty, then there is some $\varphi \in X$ such that $X \leqslant_\alpha \{\varphi\}$.*

Proof. The proof follows by induction on the site of X.

Base: $|X| = 1$. Thus, $X = \{\varphi\}$, for some $\varphi \in \mathcal{L}$. As Cn satisfies inclusion, we have that $\{\varphi\} \subseteq Cn(\{\varphi\})$. Thus, from *isotonicity*, $\{\varphi\} \leqslant_\alpha \{\varphi\}$, that is, $X \leqslant_\alpha \{\varphi\}$.

Induction Hypothesis (IH): if $Y \subseteq \mathcal{K}$ and $|Y| < |X|$ then there is some $\varphi \in Y$ such that $Y \leqslant_\alpha \{\varphi\}$

Induction Step: $|X| > 1$. Then $X = Y \cup Y'$, for some $Y, Y' \subseteq X$ such that $|Y| > 1, |Y'| > 1$, and $Y \cap Y' = \emptyset$. Note that $|Y| < |X|$, $|Y'| < |X|$. From *conjunctiveness*, $X \leqslant_\alpha Y$ or $X \leqslant_\alpha Y'$. Without loss of generality, let assume the $X \leqslant_\alpha Y$. Thus, from **IH**, that there is some $\varphi \in Y$ such that $Y \leqslant_\alpha \{\varphi\}$. Thus, from transitivity, $X \leqslant_\alpha \{\varphi\}$, and $\varphi \in X$.

□

Lemma A.3. *Given an α-shard $\leqslant_\alpha$ on a belief base $\mathcal{K}$ such that $\alpha \in Cn(\mathcal{K})$. If $\{\varphi\} \in \max_{\leqslant_\alpha}(\mathcal{P}(\mathcal{K}))$ then $A \in \max_{\leqslant_\alpha}(\mathcal{P}(\mathcal{K}))$, for all $A \subseteq \mathcal{K}$ such that $\varphi \in A$.*

Proof. Let $\leqslant_\alpha$ be an α-shard on a belief base $\mathcal{K}$ such that $\alpha \in Cn(\mathcal{K})$. Moreover, let $\varphi \in \max_{\leqslant_\alpha}(\mathcal{P}(\mathcal{K}))$, and a $A \subseteq \mathcal{K}$ such that $\varphi \in A$. From *isotonicity*, $\{\varphi\} \leqslant_\alpha A$. Therefore, as $\{\varphi\}$ is maximal, we get that $A \leqslant_\alpha \{\varphi\}$ which means that $A \in max_{\leqslant_\alpha}(\mathcal{P}(\mathcal{K}))$. □

Proposition 11. *If $\leqslant_\alpha$ is an α-shard on a belief base $\mathcal{K}$,*

1. *every α-susceptible formula w.r.t $\leqslant_\alpha$ is not α-free;*
2. *α is not tautological and $\alpha \in Cn(\mathcal{K})$ iff there is an α-susceptible formula in $\mathcal{K}$.*

Proof. Let $\leqslant_\alpha$ be an α-shard on a belief base $\mathcal{K}$.

1. let φ be an α-susceptible formula modulo an α-shard $\leqslant_\alpha$. Thus, φ does not appear in any of the resistant sets. By definition, the set of all α-free formulae is resistant. Thus, φ does not appear in such a set, which means φ is not α-free.
2. the direction "$\Leftarrow$" follows from item 1, because an α-susceptible formula φ necessarily is not α-free which implies that there is some α-kernel $A \in \mathcal{K} \perp\!\!\!\perp \alpha$ such $\varphi \in A$. Therefore, $\alpha \in Cn(\mathcal{K})$. For the direction "$\Rightarrow$", from $\alpha \in Cn(\mathcal{K})$ we get there is at least one α-kernel $X \in \mathcal{K} \perp\!\!\!\perp \alpha$, and from compactness we know that all of them are finite. Let us fix an α-kernel $X \in \mathcal{K} \perp\!\!\!\perp \alpha$. From α-*maximality*, we get that X is maximal, and from Lemma A.2, there is a $\varphi \in X$ such that $X \leqslant_\alpha \{\varphi\}$. Let us fix such a φ. Therefore, as X is maximal, we get that $\{\varphi\}$ is also maximal. This implies from Lemma A.3, that every set in which φ appears is also maximal. Therefore, every set in which φ appears is not resistant. This means that φ is α-susceptible.

□

Proposition A.4. *For every belief base $\mathcal{K}$, and formulae α and β. The following statements are equivalent:*

1. *$\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$;*
2. *for every $\mathcal{K}' \subseteq \mathcal{K}'$, $\alpha \in Cn(\mathcal{K}')$ iff $\beta \in Cn(\mathcal{K}')$.*

Proof. Let $\mathcal{K}$ be a belief base and α and β be formulae.

- "(1) $\Rightarrow$ (2)". Let us assume that $\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$, and let $\mathcal{K}' \subseteq \mathcal{K}$. We have to show that (a) if $\alpha \in Cn(\mathcal{K}')$ then $\beta \in Cn(\mathcal{K}')$; and (b) if $\beta \in Cn(\mathcal{K}')$ then $\alpha \in Cn(\mathcal{K}')$.

  (a) let $\alpha \in Cn(\mathcal{K}')$. Then there is some $X \in \mathcal{K}' \perp\!\!\!\perp \alpha$. Thus, as $X \subseteq \mathcal{K}'$ and $\mathcal{K}' \subseteq \mathcal{K}$, we get that $X \subseteq \mathcal{K}$, which means $X \in \mathcal{K} \perp\!\!\!\perp \alpha$. From hypothesis, $\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$, which implies that $X \in \mathcal{K} \perp\!\!\!\perp \beta$. This means that $\beta \in Cn(X)$. Thus, as $X \subseteq \mathcal{K}'$, and Cn is monotonic, we get that $\beta \in Cn(\mathcal{K}')$.

  (b) if $\beta \in Cn(\mathcal{K}')$. Then there is some $X \in \mathcal{K}' \perp\!\!\!\perp \beta$. Thus, as $X \subseteq \mathcal{K}'$ and $\mathcal{K}' \subseteq \mathcal{K}$, we get that $X \subseteq \mathcal{K}$, which means $X \in \mathcal{K} \perp\!\!\!\perp \beta$. From hypothesis, $\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$, which implies that $X \in \mathcal{K} \perp\!\!\!\perp \alpha$. This means that $\alpha \in Cn(X)$. Thus, as $X \subseteq \mathcal{K}'$, and Cn is monotonic, we get that $\alpha \in Cn(\mathcal{K}')$.
- "(2) $\Rightarrow$ (1)". Let us assume that for every $\mathcal{K}' \subseteq \mathcal{K}'$, $\alpha \in Cn(\mathcal{K}')$ iff $\beta \in Cn(\mathcal{K}')$. We will show that $\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$. For this we need to show that (a) $\mathcal{K} \perp\!\!\!\perp \alpha \subseteq \mathcal{K} \perp\!\!\!\perp \beta$ and (b) $\mathcal{K} \perp\!\!\!\perp \beta \subseteq \mathcal{K} \perp\!\!\!\perp \alpha$.

  (a) $\mathcal{K} \perp\!\!\!\perp \alpha \subseteq \mathcal{K} \perp\!\!\!\perp \beta$. Let $X \in \mathcal{K} \perp\!\!\!\perp \alpha$. Thus, $\alpha \in Cn(X)$ which implies from hypothesis that $\beta \in Cn(X)$. Let $X' \subset X$. Thus, as X is an α-kernel, we have that $\alpha \notin Cn(X')$, which implies from hypothesis, that $\beta \notin Cn(X')$. Therefore, $X \in \mathcal{K} \perp\!\!\!\perp \beta$.

  (b) Analogous to item (a).

□

Proposition 13. *Every effacing is an incision function.*

Proof. Let δ_τ be an effacing on a belief base $\mathcal{K}$. We need to show that δ_τ satisfies conditions (1), (2) and (3) from Definition 3. Let $\alpha \in \mathcal{L}$, and $\leqslant_\tau^\alpha$ the corresponding α-shard given by τ.

(1) We will show that $\delta_\tau(\alpha) \subseteq \bigcup \mathcal{K} \perp\!\!\!\perp \alpha$. From Proposition 11, we have that every α-susceptible formulae in $\mathcal{K}$ is not α-free, which means that $\delta_\tau(\alpha) \subseteq \bigcup \mathcal{K} \perp\!\!\!\perp \alpha$.

(2) Let $X \in \mathcal{K} \perp\!\!\!\perp \alpha$ such that $X \neq \emptyset$. We will show $X \cap \delta_\tau(\alpha) \neq \emptyset$, that is, there is some $\varphi \in X$ such that $\varphi \in \delta_\tau(\alpha)$. As X is an α-kernel, we get: (i) that X is maximal, from α-maximality; and (ii) that X is finite, from Observation A.1. The latter implies from Lemma A.2 that there is some $\varphi \in X$ such that $X \leqslant_\alpha \{\varphi\}$. Note that φ is not α-free, as X is an α-kernel. Therefore, as X is maximal and $X \leqslant_\alpha \{\varphi\}$, we also that $\{\varphi\}$ is also maximal. Therefore, from Lemma A.3, every set in which φ appears is also maximal and not α-free (because φ is not α-free). This means that every set that has φ is not resistant, which implies that φ is α-susceptible. Therefore, $\varphi \in \delta_\tau(\alpha)$.

(3) let $\beta \in \mathcal{L}$, such that $\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$. We will show that $\delta_\tau(\alpha) = \delta_\tau(\beta)$. Thus,

$$\text{for all } A \subseteq \mathcal{K}, A \text{ is not } \alpha\text{-free iff } A \text{ is not } \beta\text{-free} \quad (1)$$

From Proposition A.4, we have that for all $\mathcal{K}' \subseteq \mathcal{K}$, $\alpha \in Cn(\mathcal{K})$ iff $\beta \in Cn(\mathcal{K})$. Therefore, from *relational uniformity*, we get that $\leqslant_\alpha = \leqslant_\beta$. This means that $\max_{\leqslant_\alpha}(\mathcal{P}(\mathcal{K})) = \max_{\leqslant_\beta}(\mathcal{P}(\mathcal{K}))$ which jointly with Eq. (1), implies that

$$\mathsf{resist}_{\leqslant_\alpha}(\mathcal{K}) = \mathsf{resist}_{\leqslant_\beta}(\mathcal{K}).$$

Thus, a formula is α-susceptible iff it is β-susceptible. This implies that $\delta_\tau(\alpha) = \delta_\tau(\beta)$.

□

Lemma A.5. *Given an α-hard $\leqslant_\alpha$ on a belief base $\mathcal{K}$. If φ is α-susceptible w.r.t $\leqslant_\alpha$ and $\{\varphi\} \leqslant_\alpha \{\psi\}$ then ψ is also α-susceptible.*

Proof. Let φ be α-susceptible w.r.t $\leqslant_\alpha$ in $\mathcal{K}$, and ψ be a formula such that $\{\varphi\} \leqslant_\alpha \{\psi\}$. Let us suppose for contradiction that ψ is not α-susceptible. As φ is α-susceptible, we have $\{\varphi\} \notin \mathsf{resist}_{\leqslant_\alpha}(\mathcal{K})$, that is,

$$\{\varphi\} \in \mathsf{max}_{\leqslant_\alpha}(\{A \subseteq \mathcal{K} \mid A \text{ is not } \alpha\text{-free}\}).$$

From the contrapositive of α-discernment we have that either $\{\varphi\} \not\leqslant_\alpha \{\psi\}$ or ψ is not α-free. Thus, as by hypothesis, $\{\varphi\} \leqslant_\alpha \{\psi\}$, we get ψ is not α-free. Thus, as $\{\varphi\}$ is maximal among all not α-free sets, we get from $\{\varphi\} \leqslant_\alpha \{\psi\}$ that ψ is also maximal among all not α-free sets. That is,

$$\{\psi\} \in \mathsf{max}_{\leqslant_\alpha}(\{A \subseteq \mathcal{K} \mid A \text{ is not } \alpha\text{-free}\}). \quad (2)$$

By hypothesis, ψ is not α-susceptible. Thus, there is an $A \in \mathsf{resist}_{\leqslant_\alpha}(\mathcal{K})$, such that $\psi \in A$. Thus,

$$A \notin \mathsf{max}_{\leqslant_\alpha}(\{A \subseteq \mathcal{K} \mid A \text{ is not } \alpha\text{-free}\}). \quad (3)$$

Note that A is not not α-free, as ψ is not α-free. From isotonicity, $\{\psi\} \leqslant_\alpha A$ which implies from Eq. (2) that $A \in \mathsf{max}_{\leqslant_\alpha}(\{A \subseteq \mathcal{K} \mid A \text{ is not } \alpha\text{-free}\})$, which contradicts Eq. (3). Thus, ψ is α-susceptible. □

Theorem 15. *Every spalled kernel contraction is smooth.*

Proof. Let δ_τ be an effacing on a belief base $\mathcal{K}$, $X \subseteq \mathcal{K}$ and $\varphi \in \delta_\tau(\alpha)$ such that $\varphi \in Cn(X)$. We will show that there is some $\psi \in X$ such that $\psi \in \delta_\tau(\alpha)$. From $\varphi \in Cn(X)$, we get that there is a $X' \in X \perp\!\!\!\perp \varphi$. Let us fix such a X'. Thus, from isotonicity, we get $\{\varphi\} \leqslant_\alpha X'$, and from Lemma A.2, there is some $\psi \in X'$ such that $X' \leqslant_\alpha \{\psi\}$. From transitivity, we get $\{\varphi\} \leqslant_\alpha \{\psi\}$. Therefore, from Lemma A.5, ψ is also α-susceptible, which means that $\psi \in \delta_\tau(\alpha)$.

□

Lemma A.6. *For every smooth incision function σ on a belief base $\mathcal{K}$. If $\mathcal{K}' \subseteq \mathcal{K}$ and $\mathcal{K}' \cap \sigma(\alpha) = \emptyset$ then $Cn(\mathcal{K}') \cap \sigma(\alpha) = \emptyset$.*

Proof. Let $\mathcal{K}' \subseteq \mathcal{K}$ and $\mathcal{K}' \cap \sigma(\alpha) = \emptyset$. We have to show that for every $\varphi \in Cn(\mathcal{K}')$, $\varphi \notin \sigma(\alpha)$. Let $\varphi \in Cn(\mathcal{K}')$. As $\sigma(\alpha) \subseteq \mathcal{K}$, the case that $\varphi \notin \mathcal{K}$ is trivial. So we focus on $\varphi \in \mathcal{K}$. Thus, from the contrapositie of smoothness, we have that $\mathcal{K}' \not\subseteq \mathcal{K}$ or $\varphi \notin Cn(\mathcal{K}')$ or $\varphi \notin \sigma(\alpha)$. By hypothesis, $\mathcal{K}' \subseteq \mathcal{K}$ and $\varphi \in Cn(\mathcal{K}')$. Thus, $\varphi \notin \sigma(\alpha)$. □

Lemma 17. *For every smooth incision function and α-projection $\leqslant^\sigma_\alpha$, if $B \cap \sigma(\alpha) = \emptyset$ and $A \leqslant^\sigma_\alpha B$ then $A \cap \sigma(\alpha) = \emptyset$.*

Proof. Let us suppose for contradiction that $A \cap \sigma(\alpha) \neq \emptyset$. As $\leqslant^\sigma_\alpha$ is the least relation satisfying the condition (1)-(4) from Definition 16, we get that at least one of the following condition must be satisfied (we will get a contradiction from each of them):

1. $B \cap \sigma(\alpha) \neq \emptyset$, which is contradiction.
2. $A \subseteq Cn(B)$. From hypothesis, $A \cap \sigma(\alpha) \neq \emptyset$ which means that there is some $\varphi \in A$ such that $\varphi \in \sigma(\alpha)$. As σ is smooth and $B \cap \sigma(\alpha) = \emptyset$, we get from Lemma A.6 that $Cn(B) \cap \sigma(\alpha) = \emptyset$. Thus, as $A \subseteq Cn(B)$, we get that $\varphi \in Cn(B)$ which implies $Cn(B) \cap \sigma(\alpha) \neq \emptyset$. A contradiction.
3. A is α-free which is a contradiction, as by hypothesis $A \cap \sigma(\alpha) \neq \emptyset$.
4. both A and B are not α-free and $(A \cup B) \cap \sigma(\alpha) = \emptyset$. However, this implies that $A \cap \sigma(\alpha) = \emptyset$ which is a contradiction.

□

Proposition 18. *If an incision function σ is smooth, then every α-projection of σ satisfies: isotonicity, α-maximality, α-discernment, conjunctiveness, and transitivity.*

Proof sketch. Let $\leqslant^\sigma_\alpha$ be an α-projection of a smooth incision function σ. Note that α-maximality follows directly from condition (1), while isotonicity follows from condition (2). Item 3 puts all α-free sets as the most preferable ones, that is, the minimal ones. This jointly with Lemma 17 and Item 1 implies that each set that is not α-free is strictly less preferable then all α-free sets. Therefore, α-discernment is satisfied.

- *conjunctiveness: let $A, B \in \mathcal{P}(\mathcal{K})$. If both are are α-free, then $A \cup B$ is also α-free, which follows from (3) that $A \cup B \leqslant^\sigma_\alpha A$. Let us proceed then to the case that one of them is not α-free. Without loss of generality, let us assume that A is not α-free. As A is not α-free, we get that $A \cup B$ is also not α-free. If $(A \cup B) \cap \sigma(\alpha) \neq \emptyset$ then either $A \cap \sigma(\alpha) \neq \emptyset$ or $B \cap \sigma(\alpha) \neq \emptyset$. In either case, it follows from condition (1) that $A \cup B \leqslant^\sigma_\alpha A$ or $A \cup B \leqslant^\sigma_\alpha B$. So, only the case $(A \cup B) \cap \sigma(\alpha) = \emptyset$ remains. Thus, as both A and $A \cup B$ are not α-free, we get from condition (4) that $A \cup B \leqslant^\sigma_\alpha A$.*
- *transitivity: let $A \leqslant^\sigma_\alpha B$ and $B \leqslant^\sigma_\alpha C$. We will show that $A \leqslant^\sigma_\alpha C$. If $C \cap \sigma(\alpha) \neq \emptyset$ or A is α-free then from condition (1) and (3) we get that $A \leqslant^\sigma_\alpha C$. Let us proceed then to the case that A is not α-free and $C \cap \sigma(\alpha) = \emptyset$. As $B \leqslant^\sigma_\alpha C$ and $C \cap \sigma(\alpha) = \emptyset$, we get from Lemma 17 that $B \cap \sigma(\alpha) = \emptyset$. This implies, also from Lemma 17, that $A \cap \sigma(\alpha) = \emptyset$. Thus, $(A \cup C) \cap \sigma(\alpha) = \emptyset$, which implies from condition (4) that $A \leqslant^\sigma_\alpha C$*

□

Proof. Let σ be a smooth incision function on a belief base $\mathcal{K}$, and $\mathcal{T}_\sigma$ its shadowing. We need to show that $\mathcal{T}_\sigma$ satisfy uniformity, and that for every formula α, the α projection $\leqslant^\sigma_\alpha$ satisfy: α-maximality, α-discernment, conjunctiveness, isotonicity and transitivity. Note that α-maximality follows directly from condition (1), while *isotonicity* follows from condition (2).

- *α-discernment*: Let us suppose for contradiction that there are formulae $\varphi, \psi \in \mathcal{K}$ such that φ is α-free, $\{\psi\} \leqslant^\sigma_\alpha \{\varphi\}$ but that ψ is not α-free. As $\leqslant^\sigma_\alpha$ is the least set satisfying conditions (1)-(4) from Definition 16, then one of the following conditions hold (we will get a contradiction for each of them):

1. $\{\varphi\} \cap \sigma(\alpha) \neq \emptyset$. This is a contradiction, as φ is α-free.
2. $Cn(\psi) \subseteq Cn(\varphi)$. Thus, as ψ is not α-free, there is a $X \in \mathcal{K} \perp\!\!\!\perp \alpha$ such that $\psi \in X$. Let us fix such a X, and let $X' = X \setminus \{\psi\}$. As X is an α-kernel, it follows that $\alpha \notin Cn(X')$. Thus, from monotonicity

$$Cn(X') \cup Cn(\varphi) \subseteq Cn(X' \cup \{\varphi\})$$

From hypothesis, $Cn(\psi) \subseteq Cn(\varphi)$. Thus,

$$Cn(X') \cup Cn(\psi) \subseteq Cn(X' \cup \{\varphi\})$$

From inclusion, $X' \subseteq Cn(X')$, and $\psi \in Cn(\psi)$. Thus,

$$X' \cup \{\psi\} \subseteq Cn(X' \cup \{\varphi\}),$$

which implies from monotonicity that

$$Cn(X' \cup \{\psi\}) \subseteq Cn(Cn(X' \cup \{\varphi\})),$$

and from idempotency,

$$Cn(X' \cup \{\psi\}) \subseteq Cn(X' \cup \{\varphi\}).$$

Recall from above that $X = X' \cup \{\psi\}$, and $\alpha \in Cn(X)$, as X is an α-kernel. Therefore,

$$\alpha \in Cn(X' \cup \{\varphi\}).$$

Thus, there is some $Y \in (X' \cup \{\varphi\}) \perp\!\!\!\perp \alpha$. Thus, $\varphi \in Y$, as $\alpha \notin Cn(X')$. This means that φ is not α-free which is a contradiction.
3. $\{\psi\}$ is α-free, which contradicts our hypothesis.
4. both φ and ψ are not α-free which is also a contradiction.

So we conclude that $\leqslant^\sigma_\alpha$ indeed satisfies α-discernment.

- *conjunctiveness:* let $A, B \in \mathcal{P}(\mathcal{K})$. If both are α-free, then $A \cup B$ is also α-free, which implies from (3) that $A \cup B \leqslant^\sigma_\alpha A$. Let us proceed then to the case that one of them is not α-free. Without loss of generality, let us assume that A is not α-free. As A is not α-free, we get that $A \cup B$ is also not α-free. If $(A \cup B) \cap \sigma(\alpha) \neq \emptyset$ then either $A \cap \sigma(\alpha) \neq \emptyset$ or $B \cap \sigma(\alpha) \neq \emptyset$. In either case, it follows from condition (1) that $A \cup B \leqslant^\sigma_\alpha A$ or $A \cup B \leqslant^\sigma_\alpha B$. So, only the case $(A \cup B) \cap \sigma(\alpha) = \emptyset$ remains. Thus, as both A and $A \cup B$ are not α-free, we get from condition (4) that $A \cup B \leqslant^\sigma_\alpha A$.
- *transitivity:* let $A \leqslant^\sigma_\alpha B$ and $B \leqslant^\sigma_\alpha C$. We will show that $A \leqslant^\sigma_\alpha C$. If $C \cap \sigma(\alpha) \neq \emptyset$ or A is α-free then from condition (1) and (3) we get that $A \leqslant^\sigma_\alpha C$. Let us proceed then to the case that A is not α-free and $C \cap \sigma(\alpha) = \emptyset$. As $B \leqslant^\sigma_\alpha C$ and $C \cap \sigma(\alpha) = \emptyset$, we get from Lemma 17 that $B \cap \sigma(\alpha) = \emptyset$. This implies, also from Lemma 17, that $A \cap \sigma(\alpha) = \emptyset$. Thus, $(A \cup C) \cap \sigma(\alpha) = \emptyset$, which implies from condition (4) that $A \leqslant^\sigma_\alpha C$

□

Observation 19. *Let σ be an incision function on a belief base $\mathcal{K}$, and let α and β be two formulae. If for all $\mathcal{K}' \subseteq \mathcal{K}$, it holds that $\alpha \in Cn(\mathcal{K}')$ iff $\beta \in Cn(\mathcal{K}')$, then $\leqslant^\sigma_\alpha = \leqslant^\sigma_\beta$.*

Proof. Let $\alpha, \beta \in \mathcal{K}$ be formulae such that for all $\mathcal{K}' \subseteq \mathcal{K}$, $\alpha \in Cn(\mathcal{K}')$ iff $\beta \in Cn(\mathcal{K}')$. As σ is an incision function, we have from condition (3) of Definition 3 that $\sigma(\alpha) = \sigma(\beta)$, and from Proposition A.4 we have $\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$. We will show that $A \leqslant^\sigma_\alpha B$ iff $A \leqslant^\sigma_\beta B$.

"⇒" Let $A \leqslant^\sigma_\alpha B$. As $\leqslant^\sigma_\alpha$ is the least relation satisfying the condition (1)-(4) from Definition 16, we get that at least one of the condition (1) to (4) must be satisfied:

1. $B \cap \sigma(\alpha) \neq \emptyset$. Thus, as $\sigma(\alpha) = \sigma(\beta)$, we also have that $B \cap \sigma(\beta) \neq \emptyset$. Thus, from the same condition (1) from Definition 16, we get $A \leqslant^\sigma_\beta B$.
2. $A \subseteq Cn(B)$. Therefore, from the same condition (2) from Definition 16, we get $A \leqslant^\sigma_\beta B$.
3. A is α-free. From above, we have that $\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$, which means that a set X is α-free iff X is β-free. Therefore, A is β-free which implies from the same condition (3) in Definition 16 that $A \leqslant^\sigma_\beta B$.
4. both A and B are not α-free and $(A \cup B) \cap \sigma(\alpha) \neq \emptyset$. From above, we have that $\mathcal{K} \perp\!\!\!\perp \alpha = \mathcal{K} \perp\!\!\!\perp \beta$, which means that a set X is α-free iff X is β-free. Therefore, both A and B are not β-free. From above, $\sigma(\alpha) = \sigma(\beta)$ which implies $(A \cup B) \cap \sigma(\beta) \neq \emptyset$. Thus, from the same condition (4) in Definition 16 we have that $A \leqslant^\sigma_\beta B$.

"⇐" Analogous to direction "⇒".

□

Corollary 21. *If an incision function is smooth, then its shadowing is a spalling.*

Proof. Let σ be a smooth incision function, and $\mathcal{T}_\sigma$ be its shadowing. From Proposition 18, each α-projection $\leqslant^\sigma_\alpha$ satisfies transitivity, isotonicity, α-maximality, α-discernment, and conjunctiveness, while from Observation 19 it also satisfies relational uniformity. This means that $\leqslant^\sigma_\alpha$ is an α-shard. Therefore, $\mathcal{T}_\sigma$ is a spalling. □

Proposition 22. *If σ is a smooth incision function on a belief base $\mathcal{K}$, then for all formula α:*

$$\sigma(\alpha) = \{\varphi \in \mathcal{K} \mid \varphi \textit{ is } \alpha\textit{-susceptible modulo } \leqslant^\sigma_\alpha\}.$$

Proof. Recall that a formula φ in $\mathcal{K}$ is α-susceptible modulo a relation $\leqslant^\sigma_\alpha$ iff $\varphi \in \bigcup \mathcal{K} \perp\!\!\!\perp \alpha$ and there is no resistant set $A \in \mathsf{resist}_{\leqslant^\sigma_\alpha}(\bigcup \mathcal{K} \perp\!\!\!\perp \alpha)$ such that $\varphi \in A$.

"⇒" It is easier to prove by its contrapositive. Let us assume that $\varphi \notin \sigma(\alpha)$, and we will show that φ is not α-susceptible. The case that φ is α-free is trivial. So we focus on the case $\varphi \in \bigcup \mathcal{K} \perp\!\!\!\perp \alpha$. Thus, there is some $A \in \mathcal{K} \perp\!\!\!\perp \alpha$ such that $\varphi \in A$. Let $A' = A \setminus \sigma(\alpha)$. As σ is an incision function, we have that $A \cap \sigma(\alpha) \neq \emptyset$. Thus, from the contrapositive of Lemma 17 (on the statement of that lemma let B stand for A') that $A \not\leqslant^\sigma_\alpha A'$ or $A' \cap \sigma(\alpha) \neq \emptyset$. However, as $A' = A \setminus \sigma(\alpha)$, we have that $A' \cap \sigma(\alpha) = \emptyset$. Therefore, $A \not\leqslant^\sigma_\alpha A'$. From condition (1), at Definition 16, we get that $A' \leqslant^\sigma_\alpha A$. Thus, A' is a resistant set which implies that φ is not α-susceptible, as $\varphi \in A'$.

"⇐" Let $\varphi \in \sigma(\alpha)$. Then, $\varphi \in \bigcup \mathcal{K} \perp\!\!\!\perp \alpha$, and it follows from condition (1) at Definition 16 that every set $A \in \mathcal{P}(\mathcal{K})$ that has φ is maximal, that is, $A \in max_{\leqslant^{\sigma}_{\alpha}}(\mathcal{P}(\mathcal{K}))$. Therefore, φ is α-susceptible.

□

Theorem 23. *A kernel contraction is smooth iff its a spalled kernel contraction.*

Proof. From Theorem 15, every spalled kernel contraction is a smooth kernel contraction. So we are left to prove direction "⇒". Let $\dot{-}_\sigma$ be a smooth kernel contraction on a belief base $\mathcal{K}$. To show that $\dot{-}_\sigma$ is a spalled kernel contraction, it suffices to show that there exist an effacing δ_τ such that $\sigma(\alpha) = \delta_\tau(\alpha)$, for all formula α. Consider the shadowing $\mathcal{T}_\sigma$ of σ. From Corollary 21, $\mathcal{T}_\sigma$ is a spalling. Let us take then the effacing $\delta_{\mathcal{T}_\sigma}$. Recall from the definition of shadowing (Definition 20) that $\mathcal{T}_\sigma(\alpha) = \leqslant^{\sigma}_{\alpha}$. By definition of effacing,

$$\delta_{\mathcal{T}_\sigma}(\alpha) = \{\varphi \in \mathcal{K} \mid \varphi \text{ is } \alpha\text{-susceptible modulo } \leqslant^{\sigma}_{\alpha}\}$$

Moreover, from Proposition 22,

$$\sigma(\alpha) = \{\varphi \in \mathcal{K} \mid \varphi \text{ is } \alpha\text{-susceptible modulo } \leqslant^{\sigma}_{\alpha}\}.$$

Therefore, $\delta_{\mathcal{T}_\sigma}(\alpha) = \sigma(\alpha)$, for all formula α. Thus, $\dot{-}_\sigma$ is a spalled kernel contraction.

□

Proposition 26. *If a kernel contraction function $\dot{-}_\sigma$ satisfies relevance then σ satisfies symmetry of removal.*

Proof. Let $\dot{-}_\sigma$ be a kernel contraction function, on a belief base $\mathcal{K}$, that satisfies relevance. Let α be a formula, and $A, B \subseteq \mathcal{K}$ be α-concordant sets. To show that σ satisfies symmetry of removal, we have to show that: (i) $A \cap \sigma(\alpha) \neq \emptyset$ iff (ii) $B \cap \sigma(\alpha) \neq \emptyset$.

"(i) ⇒ (ii)". Let us suppose for contradiction that $A \cap \sigma(\alpha) \neq \emptyset$ but $B \cap \sigma(\alpha) = \emptyset$. Thus, there is some $\varphi \in A$, such that $\varphi \in \sigma(\alpha)$. From relevance, there is a $\mathcal{K}' \subseteq \mathcal{K}$ such that $\mathcal{K} \dot{-}_\sigma \alpha \subseteq \mathcal{K}'$ and

$$\alpha \notin Cn(\mathcal{K}') \text{ and } \alpha \in Cn(\mathcal{K}' \cup \{\varphi\})$$

As $\varphi \in A$, we have $\mathcal{K}' \cup \{\varphi\} \subseteq A \cup \mathcal{K}'$. Thus, as Cn is monotonic, we have $Cn(\mathcal{K}' \cup \{\varphi\}) \subseteq Cn(A \cup \mathcal{K}')$. Thus, as from above $\alpha \in Cn(\mathcal{K}' \cup \{\varphi\})$, we get that $\alpha \in Cn(A \cup \mathcal{K}')$. This means $\mathcal{K}'$ is an α-completion of A, that is

$$\mathcal{K}' \in com_{\mathcal{K}}(A, \alpha)$$

As $B \cap \sigma(\alpha) = \emptyset$, we have $B \subseteq \mathcal{K} \dot{-}_\sigma \alpha$, which implies that $B \subseteq \mathcal{K}'$. Therefore, $B \cup \mathcal{K}' = \mathcal{K}'$ which means $Cn(\mathcal{K}') = Cn(B \cup \mathcal{K}')$. Thus, as $\alpha \notin Cn(\mathcal{K}')$, we get $\alpha \notin Cn(B \cup \mathcal{K}')$. This means that $\mathcal{K}' \notin com_{\mathcal{K}}(B, \alpha)$. Therefore, A and B are not α-concordant which contradicts our hypothesis. Thus, $B \cap \sigma(\alpha) \neq \emptyset$.

"(ii) ⇒ (i)". Analogous to "(i) ⇒ (ii)".

□

Lemma A.7. *For every belief base $\mathcal{K}$, and sets $A, B \subseteq \mathcal{K}$, if $A \subseteq B$ then $com_{\mathcal{K}}(A, \alpha) \subseteq com_{\mathcal{K}}(B, \alpha)$.*

Proof. Let $A \subseteq B$, and $X \in com_{\mathcal{K}}(A, \alpha)$. Then, $\alpha \in Cn(X \cup A)$. As $A \subseteq B$ we have that $X \cup A \subseteq X \cup B$ which from monotonicity of Cn implies $Cn(X \cup A) \subseteq Cn(X \cup B)$. Thus, as $\alpha \in Cn(X \cup A)$, we get that $\alpha \in Cn(X \cup B)$. Thus, $X \in com_{\mathcal{K}}(B, \alpha)$. □

Proposition 28. *If an incision function σ satisfies smoothness and symmetry of removal then the smooth kernel contraction function $\dot{-}_\sigma$ satisfies relevance.*

Proof. Let $\dot{-}_\sigma$ be a smooth kernel contraction on a belief base $\mathcal{K}$, and formulae α and β such that $\beta \in \mathcal{K} \setminus (\mathcal{K} \dot{-}_\sigma \alpha)$. To prove relevance, we will show that there is a belief base $\mathcal{K}'$ such that (a) $\mathcal{K} \dot{-}_\sigma \alpha \subseteq \mathcal{K}' \subseteq \mathcal{K}$, (b) $\alpha \notin Cn(\mathcal{K}')$, and (c) $\alpha \in Cn(\mathcal{K}' \cup \{\beta\})$.

From $\beta \in \mathcal{K} \setminus (\mathcal{K} \dot{-}_\sigma \alpha)$, we get that $\beta \in \sigma(\alpha)$. Let

$$Y = (\mathcal{K} \dot{-}_\sigma \alpha) \cup \{\beta\}.$$

Note $\mathcal{K} \dot{-}_\sigma \alpha \cap \sigma(\alpha) = \emptyset$, but $Y \cap \sigma(\alpha) \neq \emptyset$. Thus, from the contrapositive of symmetry of removal, we get that $\mathcal{K} \dot{-}_\sigma \alpha$ and Y are not α-concordant. As $\mathcal{K} \dot{-}_\tau \alpha \subseteq Y$ we have from Lemma A.7 that $com_{\mathcal{K}}(\mathcal{K} \dot{-}_\sigma \alpha, \alpha) \subseteq com_{\mathcal{K}}(Y, \alpha)$. Moreover, as $\mathcal{K} \dot{-}_\sigma \alpha$ and Y are not α-concordant, we have that $com_{\mathcal{K}}(\mathcal{K} \dot{-}_\sigma \alpha, \alpha) \neq com_{\mathcal{K}}(Y, \alpha)$. Therefore, $com_{\mathcal{K}}(\mathcal{K} \dot{-}_\sigma \alpha, \alpha) \subset com_{\mathcal{K}}(Y, \alpha)$, which implies that there is some $H \in com_{\mathcal{K}}(Y, \alpha)$, such that $H \notin com_{\mathcal{K}}(\mathcal{K} \dot{-}_\sigma \alpha, \alpha)$. Thus,

$$\alpha \in Cn(Y \cup H), \text{ and } \alpha \notin Cn(\mathcal{K} \dot{-}_\tau \alpha \cup H)$$

Let us fix such a H. Thus, as $Y = (\mathcal{K} \dot{-}_\sigma \alpha) \cup \{\beta\}$, we get

$$\alpha \in Cn((\mathcal{K} \dot{-}_\sigma \alpha) \cup \{\beta\} \cup H)$$

Let us make $\mathcal{K}' = (\mathcal{K} \dot{-}_\sigma \alpha) \cup H$. Thus,

$$(c)\ \alpha \in Cn(\mathcal{K}' \cup \{\beta\}), \text{ and } (b)\ \alpha \notin Cn(\mathcal{K}').$$

As $\mathcal{K} \dot{-}_\tau \alpha \subseteq \mathcal{K}$ and $H \subseteq \mathcal{K}$: (a) $\mathcal{K} \dot{-}_\tau \alpha \subseteq \mathcal{K}' \subseteq \mathcal{K}$. □

Lemma A.8. *For every belief base $\mathcal{K}$ and sets $A, X \subseteq \mathcal{K}$, if A is α-free and $X \in \mathcal{K} \perp\!\!\!\perp \alpha$ but $A \cap X = \emptyset$, then $\alpha \notin Cn(Y \cup A)$ for all $Y \subset X$.*

Proof. Let us suppose for contradiction that A is α-free and $X \in \mathcal{K} \perp\!\!\!\perp \alpha$ but $A \cap X = \emptyset$, but there is a $Y \subset X$ such that $\alpha \in Cn(Y \cup A)$. Thus, there is some $Y' \in (Y \cup A) \perp\!\!\!\perp \alpha$. Thus,

$$\alpha \in Cn(Y')$$

As $Y' \subseteq Y \cup A$ and $Y \cup A \subseteq \mathcal{K}$, we have $Y' \subseteq \mathcal{K}$ which implies $Y' \in \mathcal{K} \perp\!\!\!\perp \alpha$. Thus, as A is α-free, we get that $A \cap Y' = \emptyset$. Therefore, as $Y' \subseteq Y \cup A$, we get $Y' \subseteq Y$ which implies $Cn(Y') \subseteq Cn(Y)$. However, as X is an α-kernel, and $Y \subset X$, we have that $\alpha \notin Cn(Y)$. This implies that $\alpha \notin Cn(Y')$, which is a contradiction.

□

Observation A.9. *If two sets $A, B \subseteq \mathcal{K}$ are α-concordant then either (i) both A and B are α-free or (ii) both A and B are not α-free.*

Proof. Let us suppose for contradiction that for some belief base $\mathcal{K}$ and formula α, there are sets $A, B \subseteq \mathcal{K}$ such that A and B are α-concordant, but conditions (i) and (ii) are not satisfied. Without loss of generality, let us assume that A is α-free, and B is not α-free. Then, there is some $X \in \mathcal{K} \perp\!\!\!\perp \alpha$, such that $B \cap X \neq \emptyset$. Let $X' = X \setminus B$. Note that $X' \subset X$ and $\alpha \in Cn(X' \cup B)$. Thus $X' \in com_{\mathcal{K}}(B, \alpha)$. As A is α-free, we have that $\alpha \notin Cn(A)$ and $X \cap A = \emptyset$ which implies Lemma A.8 that $\alpha \notin Cn(X' \cup A)$. Thus, $X' \notin com_{\mathcal{K}}(A, \alpha)$. But then, as A and B are α-concordant, we get that $X' \notin com_{\mathcal{K}}(B, \alpha)$ which is a contradiction. □

Proposition A.10. *Every mirrored effacing satisfies symmetry of removal.*

Proof. Let us suppose for contradiction that there is a mirrored effacing δ_τ, defined on some belief base $\mathcal{K}$, that does not satisfy symmetry of removal. Thus there are α-concordant sets $A, B \subseteq \mathcal{K}$ such that it does not hold that $A \cap \delta_\tau(\alpha) \neq \emptyset$ iff $B \cap \delta_\tau(\alpha) \neq \emptyset$. Without loss of generality, let us assume that $A \cap \delta_\tau(\alpha) = \emptyset$ and $B \cap \delta_\tau(\alpha) \neq \emptyset$. This means that A is α-resistant modulo $\leqslant^\tau_\alpha$ while B is not α-resistant modulo $\leqslant^\tau_\alpha$. From Observation A.9, either (i) both A and B are α-free or (ii) both A and B are not α-free. For case (i), both A and B are by definition α-resistant, which is a contradiction. So we focus on case (ii). As A is resistant, but not α-free, it follows that there is an $X \subseteq \mathcal{K}$ such that $A \leqslant^\tau_\alpha X$ and $X \not\leqslant^\tau_\alpha A$. Let us fix such an X. Let $Y \in \mathcal{K} \perp\!\!\!\perp \alpha$. Thus, from α-maximality, we get that $X \leqslant^\tau_\alpha Y$ and $B \leqslant^\tau_\alpha Y$. As B is not resistant, we also get that $Y \leqslant^\tau_\alpha B$. From transitivity, we have that $X \leqslant^\tau_\alpha B$ which from concordant-mirroring implies that $X \leqslant^\tau_\alpha A$, which is a contradiction.

□

Theorem 32. *Mirrored effacings satisfy symmetry of removal, and every mirrored kernel contraction satisfies relevance.*

Proof. We have to prove: that (1) Every mirrored effacing satisfies symmetry of removal; and (2) very mirrored kernel contraction satisfies relevance. Item (1) is Proposition A.10. So we only need to prove item (2). Let $\dot{-}_\tau$ be a mirrored kernel contraction, and δ_τ its mirrored effacing. Thus, from Proposition A.10, δ_τ satisfies the principle of symmetry of removal. From Theorem 15, every effacing satisfies smoothness. Thus, from Proposition 28, we have $\dot{-}_\tau$ satisfies relevance. □

Proposition 33. *If a smooth kernel contraction function $\dot{-}_\sigma$ satisfies relevance then the shadowing of σ is mirrored.*

Proof. Let $\dot{-}_\sigma$ be a smooth kernel contraction satisfying relevance, and $\mathcal{T}_\sigma$ be the shadowing of σ. To show that $\mathcal{T}_\sigma$ is mirrored we need to show that $\leqslant^\sigma_\alpha$ satisfies concordant-mirorring for all formula α. Let α be a formula, and let $A, B \subseteq \mathcal{K}$ be α-concordant sets, and $X \leqslant^\sigma_\alpha A$. We will show that $X \leqslant^\sigma_\alpha B$. As $\dot{-}_\tau$ satisfies relevance, it follows from Proposition 26 that $\dot{-}_\tau$ satisfies the principle of symmetry of removal. So, $A \cap \sigma(\alpha) \neq \emptyset$ iff $B \cap \sigma(\alpha) \neq \emptyset$. We have two cases: either (a) $A \cap \sigma(\alpha) \neq \emptyset$ or (b) $A \cap \sigma(\alpha) = \emptyset$.

(a) $A \cap \sigma(\alpha) \neq \emptyset$. Thus, from symmetry of removal, $A \cap \sigma(\alpha) \neq \emptyset$. Thus, from Item 1 at Definition 16, we have $X \leqslant^\sigma_\alpha B$.

(b) $A \cap \sigma(\alpha) = \emptyset$. Thus, from symmetry of removal, $A \cap \sigma(\alpha) = \emptyset$, which implies that $(A \cup B) \cap \sigma(\alpha) = \emptyset$. From Observation A.9 either: (i) both A and B are α-free, or (ii) both are not α-free. For case (i) we get from Item 3 at Definition 16 that $A \leqslant^\sigma_\alpha B$; while for case (ii) we get from Item 4 at Definition 16, that $A \leqslant^\sigma_\alpha B$. So, in either case $A \leqslant^\sigma_\alpha B$. From hypothesis $X \leqslant^\sigma_\alpha A$, which from transitivity of $\leqslant^\sigma_\alpha$ implies that $X \leqslant^\sigma_\alpha B$.

□

Theorem 34. *A smooth kernel contraction satisfies relevance iff its a mirrored kernel contraction.*

Proof. From Theorem 32, every mirrored kernel contraction satisfies relevance. For the other direction, let $\dot{-}_\sigma$ be a smooth kernel contraction function satisfying relevance.

To show that $\dot{-}_\sigma$ is a mirrored kernel contraction, it suffices to show that there exist an effacing δ_τ such that (i) τ is mirrored, and (ii) $\sigma(\alpha) = \delta_\tau(\alpha)$, for all formula α. Consider the shadowing $\mathcal{T}_\sigma$ of σ. From Corollary 21, $\mathcal{T}_\sigma$ is a spalling. Let us take then the effacing $\delta_{\mathcal{T}_\sigma}$. From Proposition 33, $\mathcal{T}_\sigma$ is mirrored (i). For condition (ii) , recall from the definition of shadowing (Definition 20) that $\mathcal{T}_\sigma(\alpha) = \leqslant^\sigma_\alpha$. By definition of effacing,

$$\delta_{\mathcal{T}_\sigma}(\alpha) = \{\varphi \in \mathcal{K} \mid \varphi \text{ is } \alpha\text{-susceptible modulo } \leqslant^\sigma_\alpha\}$$

Moreover, from Proposition 22,

$$\sigma(\alpha) = \{\varphi \in \mathcal{K} \mid \varphi \text{ is } \alpha\text{-susceptible modulo } \leqslant^\sigma_\alpha\}.$$

Therefore, $\delta_{\mathcal{T}_\sigma}(\alpha) = \sigma(\alpha)$, for all formula α. Thus, $\dot{-}_\sigma$ is a mirroed kernel contraction.

□